8.2 L'Hôpital

Guillaume François Antoine de l'Hôpital (1661–1704)

Marquis de St. Mesme

Limits

$\lim_{x\to 0} \frac{K}{x} = \infty \qquad \lim_{x\to\infty} \frac{K}{x} = 0$

$\lim_{x\to 0} \frac{x}{k} = 0 \qquad \lim_{x\to\infty} \frac{x}{k} = \infty$

$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \text{"}\frac{0}{0}\text{"}$

$\lim_{x\to 0} \frac{\sin x}{x} = \text{"}\frac{0}{0}\text{"} = 1$

L'Hôpital's Rule

If $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(a)}{g(a)} = \text{"}\frac{0}{0}\text{"} = \frac{f'(a)}{g'(a)}$

(as long as $f'(a)$ and $g'(a)$ exist and $g'(a) \neq 0$)

Why?

$$\frac{f'(a)}{g'(a)} = \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} = \lim_{x\to a} \frac{f(x)-\cancel{f(a)}^{0}}{g(x)-\cancel{g(a)}^{0}}$$

$$= \lim_{x\to a} \frac{f(x)}{g(x)}$$

① $\lim_{x\to 0} \frac{\sin x}{x} = \text{"}\frac{0}{0}\text{"} = \lim_{x\to 0} \frac{\cos x}{1} = \frac{1}{1} = 1$

② $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \text{"}\frac{0}{0}\text{"} = \lim_{x\to 0} \frac{\sin x}{2x} = \text{"}\frac{0}{0}\text{"}$

extra-strength L'Hopital: $\lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$

One-Sided Limits

$$\lim_{x\to 0^+} \frac{\sin x}{x^2} = \text{"}\frac{0}{0}\text{"} = \lim_{x\to 0^+} \frac{\cos x}{2x} \ \text{"}\frac{1}{0}\text{"} = \infty$$

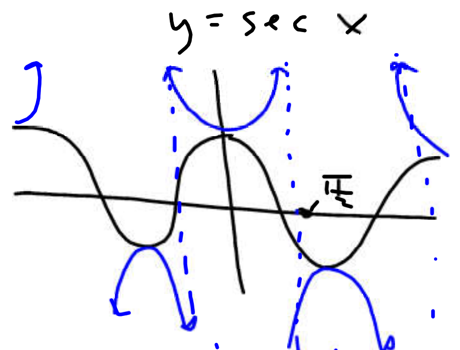
$$\lim_{x\to 0^-} \frac{\sin x}{x^2} = \text{"}\frac{0}{0}\text{"} = \lim_{x\to 0^-} \frac{\cos x}{2x} = \text{"}\frac{1}{0}\text{"} = -\infty$$

Also works for $\frac{\infty}{\infty}$

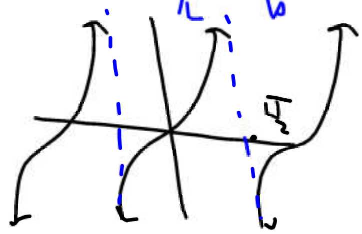
1. $$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \text{"}\frac{\infty}{\infty}\text{"} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}} =$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$

2. $$\lim_{x\to\frac{\pi}{2}} \frac{\sec x}{1+\tan x}$$

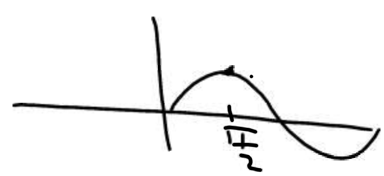


$$\lim_{x\to\frac{\pi}{2}^-} \frac{\sec x}{1+\tan x} = \frac{\infty}{\infty}$$



$$\lim_{x\to\frac{\pi}{2}^-} \frac{\sec x\tan x}{\sec^2 x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\tan x}{\sec x}$$

$$\lim_{x\to\frac{\pi}{2}^-} \sin x = 1$$



On right:

$$\lim_{x\to\frac{\pi}{2}^+} \frac{\sec x}{1+\tan x} = \frac{-\infty}{-\infty} = \lim_{x\to\frac{\pi}{2}^+} \sin x = 1$$

LH + RH same, so $\displaystyle\lim_{x\to\frac{\pi}{2}} \frac{\sec x}{1+\tan x} = 1$

Dealing with $0 \cdot \infty$ and $\infty - \infty$

Side note: $\infty \cdot \infty = \infty$ $\qquad 0 \cdot 0 = 0$

$$\lim_{x\to\infty} x^2 = \infty \qquad \lim_{x\to 0} x^2 = 0$$

$$0 - 0$$

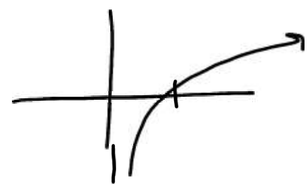
$$\lim_{x\to 0} x^2 - x = 0$$

① $\lim_{x\to\infty} \left(x \cdot \sin \frac{1}{x}\right) = \infty \cdot 0$ $\quad \left(\frac{1}{x} \to 0\right)$

$$= \lim_{x\to\infty} \left(\frac{\sin \frac{1}{x}}{\frac{1}{x}}\right) = \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{x\to\infty} \frac{\cos \frac{1}{x} \cdot \frac{-1}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \cos \frac{1}{x} = 1 \quad \left(\frac{1}{x} \to 0\right)$$

② $\lim\limits_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) = \infty - \infty$



Make it <u>one</u> fraction:

$$\lim_{x\to 1}\left(\frac{x-1-\ln x}{(\ln x)(x-1)}\right) = \frac{0}{0} = \lim_{x\to 1}\frac{1-\frac{1}{x}}{\ln x\cdot 1+(x-1)\frac{1}{x}}$$

$$\lim_{x\to 1}\frac{x-1}{x\ln x+(x-1)} = \frac{0}{0} = \lim_{x\to 1}\frac{1}{x\cdot\frac{1}{x}+\ln x\cdot 1+1}$$

$$= \lim_{x\to 1}\frac{1}{1+\ln x+1} = \frac{1}{2}$$

Tomorrow ... $1^{\infty}, 0^{0}, \infty^{0}$

HW: p450 #1-8 all, 9-19 odds